ON THE INSTABILITY OF THE AXIS OF THE GYROSCOPE FIGURE

(O NEUSTOICHIVOSTI OSI FIGURY GIROSCOPA)

PMM Vol.24, No.4, 1960, pp. 763-765

Ia.L. LUNTS
(Leningrad)

(Received 2 February 1960)

By using the first integrals of the equations of motion of a gyroscope on gimbals it is shown in this paper that an arbitrary small perturbation of the inner ring causes a precessional motion of the outer ring, which displaces the rotor axis from its initial position.

Rumiantsev [7,8], and Skimel [10] have constructed the Liapunov functions and given the conditions for stability of the regular precession and for the permanent rotation of a heavy gyroscope. In the above investigations the case of permanent rotation of an equilibrated gyroscope about an arbitrary axis has been omitted.

By the use of the equations of motion and the resulting quadratures of a gyroscopic system [1] as obtained by Chetaev [9] and Skimel [10] the instability of the axis of the gyroscope figure can be demonstrated. The equations of motion have the form

$$\dot{\psi} = -\frac{J\Omega_0 (\sin\vartheta - \sin\vartheta_0)}{A - B\sin^2\vartheta}, \qquad \dot{\vartheta}^2 = \dot{\vartheta}_0{}^2 - \frac{J^2\Omega_0{}^2 (\sin\vartheta - \sin\vartheta_0)^2}{J_B(A - B\sin^2\vartheta)} \qquad (1)$$

Here ψ is the rotation angle of the outer ring, ϑ is the rotation angle of the inner ring about its axis of rotation (this angle is measured with respect to coordinates fixed in the outer ring), J is the moment of inertia of the rotor about its spin axis, Ω_0, $\dot{\vartheta}_0$ and ϑ_0 are the initial values of the angular velocity of the rotor, angular velocity of the outer ring and the rotation angle of the inner ring (this angle is measured from such position of the inner ring when the spin axis is perpendicular to the outer ring).

$$A = J_z^{(1)} + J_z^{(2)} + J_z, \qquad B = J_z^{(2)} + J_z - J_x^{(2)}, \qquad J_B = J_y^{(2)} + J_z \qquad (2)$$

Here $J_z^{(1)}$ is the moment of inertia of the outer ring about its axis of rotation, $J_x^{(2)}$, $J_y^{(2)}$ and $J_z^{(2)}$ are moments of inertia of the inner ring with respect to the x-, y- and z-axes forming an orthogonal system where the x-axis coincides with the spin axis and the y-axis with the axis of rotation of the inner ring, and J_z is the equatorial moment of inertia of the rotor.

It is easily seen that for an arbitrary value of ϑ

$$J_C(\vartheta) = A - B\sin^2\vartheta > 0, \qquad J_B > 0 \tag{3}$$

The second equation in (1) is the equation of the family of "phase" trajectories in the phase plane (ϑ, $\dot{\vartheta}$) with parameters $\vartheta_0, \dot{\vartheta}_0$. All these phase trajectories are symmetric with respect to the axis $\dot{\vartheta} = 0$, for if (ϑ, $\dot{\vartheta}$) is a point on the trajectory, so is the point ($\vartheta, -\dot{\vartheta}$).

Fixing ϑ_0 and varying $\dot{\vartheta}_0$, we obtain a one-parameter family of trajectories. The point $\vartheta = \vartheta_0$, $\dot{\vartheta} = 0$ is the center. In a sufficiently small neighborhood of the position of equilibrium all phase trajectories are closed. Indeed, we find now points of intersection of a phase trajectory with the axis $\dot{\vartheta} = 0$.

To obtain them we shall set $\dot{\vartheta} = 0$ in the second equation of (1) and solve the resulting equation for $\sin\vartheta$; we have then

$$\sin\vartheta_{2,1} = \frac{\sin\vartheta_0 \pm \mu\sqrt{A - B\sin^2\vartheta_0 + AB\mu^2}}{1 + B\mu^2} \qquad \left(\mu = \frac{\dot{\vartheta}_0\sqrt{J_B}}{J\Omega_0}\right) \tag{4}$$

The condition for the existence of real roots ϑ_1 and ϑ_2 of Equation (3) is given by the following inequalities:

$$1 \leqslant \sin\vartheta_1 \leqslant 1, \ -1 \leqslant \sin\vartheta_2 \leqslant 1.$$

It is seen from Expression (4) that when $\vartheta_0 = 1/2\,\pi$ and $\dot{\vartheta}_0$ is sufficiently small (which means that μ is also small) then both inequalities are satisfied.

In this way, to each sufficiently small value of the initial velocity $\dot{\vartheta}_0$ corresponds a closed phase trajectory intersecting the axis $\dot{\vartheta} = 0$ at two points ϑ_1 and $\vartheta_2\,(\vartheta_1 < \vartheta_0 < \vartheta_2)$, which are determined from Equation (4).

The closed phase trajectories correspond to periodic solutions of the second equation in (1). The initial conditions $\vartheta_0 = \dot{\vartheta}_0 = 0$ correspond to the position of equilibrium $\vartheta \equiv \vartheta_0$, $\psi \equiv \psi_0$.

Thus, in a certain neighborhood of the center, all solutions $\vartheta(t)$ are periodic. The function is also periodic, because if we substitute on the right-hand side of the first equation in (1) the periodic function $\vartheta(t)$, then this right-hand side must also be a periodic function of time.

We shall find now the increment of the angle ψ in one period.

If T is the period, then obviously

$$\Delta\psi = \int_0^T \dot{\psi}\,dt$$

Passing from the variable t to the variable ϑ and utilizing the second equation of (1) we obtain [9,10]

$$dt = \frac{\operatorname{sign}\dot{\vartheta})\,\sqrt{J_B}\,\sqrt{A - B\sin^2\vartheta}\,d\vartheta}{J\Omega_0\sqrt{(A - B\sin^2\vartheta)\,\mu^2 - (\sin\vartheta - \sin\vartheta_0)^2}}.$$

The integration with respect to ϑ should be carried out from ϑ_1 to ϑ_2 when sign $\dot{\vartheta} = 1$ (ϑ is increasing), and from ϑ_2 to ϑ_1 when sign $\dot{\vartheta} = -1$.

In this way we have

$$\Delta\psi = 2\sqrt{J_B}\int_{\vartheta_1}^{\vartheta_2} \frac{(\sin\vartheta - \sin\vartheta_0)\,d\vartheta}{\sqrt{(A - B\sin^2\vartheta)\,[(A - B\sin^2\vartheta)\,\mu^2 - (\sin\vartheta - \sin\vartheta_0)^2]}}$$

We notice that

$$(A - B\sin^2\vartheta)\,\mu^2 - (\sin\vartheta - \sin\vartheta_0)^2 = (1 + B\mu^2)(\sin\vartheta_2 - \sin\vartheta)(\sin\vartheta - \sin\vartheta_1)$$

Hence

$$\Delta\psi = -2\sqrt{\frac{J_B}{1 + B\mu^2}}\int_{\vartheta_1}^{\vartheta_2} \frac{(\sin\vartheta - \sin\vartheta_0)\,d\vartheta}{\sqrt{(A - B\sin^2\vartheta)(\sin\vartheta - \sin\vartheta_1)(\sin\vartheta_2 - \sin\vartheta)}} \tag{6}$$

The elementary proof of the existence of the improper integral in (6) is omitted, but we shall prove that the integral in (6) does not vanish when $\mu \neq 0$ and $\sin\vartheta_0 \neq 0$.

We shall perform the following change of variables:

$$\sin t = \frac{\sin\vartheta - z_1}{z_0} \tag{7}$$

Here

$$z_1 = \frac{\sin\vartheta_1 + \sin\vartheta_2}{2} = \frac{\sin\vartheta_0}{1 + B\mu^2}, \quad z_0 = \frac{\sin\vartheta_2 - \sin\vartheta_1}{2} = \frac{\mu\sqrt{A - B\sin^2\vartheta_0 + AB\mu^2}}{1 + B\mu^2}$$

Then Formula (6) becomes

$$a(\mu) \equiv \Delta\psi = -2\mu \sqrt{J_B(1+B\mu^2)}\, b(\mu), \qquad b(\mu) = \int_{-1/2\pi}^{1/2\pi} F(\mu, t)\, dt \tag{8}$$

$$F(\mu, t) = \frac{\sqrt{J_C + AB\mu^2}\sin t - B\mu \sin\vartheta_0}{\theta(t;\, A, B)\,\theta(t;\, 1,1)}$$
$$\theta(t;\, a, b) = [a(1 + b\mu^2)^2 - b(\mu\sqrt{J_C + AB\mu^2}\sin t + \sin\vartheta_0)^2]^{1/2} \tag{9}$$

We shall assume that $a(0) = \lim a(\mu) = 0$ when $\mu \to 0$. To prove that in a certain neighborhood of the point $\mu = 0$ $a(\mu) \neq 0$, it is sufficient to show that the function $b(\mu)$ which is continuously differentiable with respect to μ has at $\mu = 0$ a nonzero derivative. Differentiation is permissible because the function $F(\mu, t)$ is analytic with respect to μ, and all its derivatives with respect to t are continuous in a certain neighborhood of $\mu = 0$.

Simple calculations give

$$b(0) = 0, \qquad b'(0) = \frac{\pi(A - B)\sin\vartheta_0}{2\cos^3\vartheta_0 \sqrt{A - B\sin^2\vartheta_0}}$$

In this way the expansion of $a(\mu)$ in Taylor series

$$a(\mu) = a_0 + a_1\mu + a_2\mu^2 + \dots$$

begins from the third term, and

$$a_2 = -\frac{\pi\sqrt{J_B}\,(A - B)\sin\vartheta_0}{\cos^3\vartheta_0 \sqrt{A - B\sin^2\vartheta_0}} \tag{10}$$

Thus, when $\mu \neq 0$, the increment $\Delta\psi$ of the rotation angle of the outer ring in one period T of nutational vibrations of the inner ring does not vanish. It means that the gyroscope on gimbals is unstable.

The axis of the figure performs a precession about the outer ring with a mean angular velocity $\dot{\psi} = \Delta\psi / T$.

From the second equation of (1) follows that the period of the nutational vibrations is

$$T = \frac{2\sqrt{J_B}}{J\Omega_0\sqrt{1 + B\mu^2}} \int_{\vartheta_1}^{\vartheta_2} \frac{\sqrt{A - B\sin^2\vartheta}\, d\vartheta}{\sqrt{(\sin\vartheta - \sin\vartheta_1)(\sin\vartheta_2 - \sin\vartheta)}}$$

After changing the variables as indicated in (7) and expanding T in the series $T = T_0 + T_1\mu + \dots$ we obtain

$$T_0 = \frac{2\pi\sqrt{J_B(A - B\sin^2\vartheta_0)}}{J\Omega_0\cos\vartheta_0}$$

Thus, the expansion of $\dot{\psi}_0$ in Taylor series

$$\dot{\psi}_0(\mu) = b_0 + b_1\mu + b_2\mu^2 + \ldots$$

begins with the term $b_2\mu^2$, so that

$$\dot{\psi}_0 = -\frac{J_B\,(J_z^{(1)} + J_x^{(2)})\,\dot{\vartheta}_0^2\,\sin\vartheta_0}{2JJ_C\,(\vartheta_0)\,\Omega_0\cos^2\vartheta_0} + \ldots$$

Formulas derived by approximate methods in [2-6] are the same as the first term of this expansion.

BIBLIOGRAPHY

1. Nikolai, E.L., *Teoria giroskopov (Theory of Gyroscopes)*. Gostekhizdat, 1948.

2. Magnus, K., Beitrage zur Dynamik des kraftefreien kardanisch gelagerten Kreisels. *ZAMM* Vol. 35, Nos. 1-2, Jan.-Febr. 1955.

3. Plymalby, B.T. and Goodstein, R., Nutation of free gyro subjected to an impulse. *J. Appl. Mech.* Sept. 22(3), 1955.

4. Pavlov, V.A., O vliianii nutatsionnykh kolebanii giroskopa na ego sistematicheskii ukhod ot zadannogo napravlenia (On the influence of nutational vibrations of a gyroscope on its systematic displacement from the assigned direction). *Tr. LIAP* Vol. 19, 1958.

5. Pel'por, D.S., Svobodnoe dvizhenie giroskopa zakliuchennogo v kardanovom podvese (Free motion of a gyroscope on gimbals). *N. Dok. V.S. Mash. i Prib.* No. 3, 1958.

6. Lunts, Ia.L., O dvizhenii po inertsii giroskopa v kardanovom podvese (On the inertial motion of a gyroscope on gimbals). *Izv. vyssh. ucheb. zaved. M.V.O. SSSR* No. 3, 1959.

7. Rumiantsev, V.V., Ob ustoichivosti dvizhenia giroskopa v kardanovom podvese (On the stability of a gyroscope on gimbals). *PMM* Vol. 22, No. 3, 1958.

8. Rumiantsev, V.V., Ob ustoichivosti dvizhenia giroskopa v kardanovom podvese II (On the stability of a gyroscope on gimbals). *PMM* Vol. 22, No. 4, 1958.

9. Chetaev, N.G., O giroskope v kardanovom podvese (On a gyroscope on gimbals). *PMM* Vol. 22, No. 3, 1958.

10. Skimel, V.N., Nekotorye zadachi dvizhenia i ustoichivosti tiazhelogo giroskopa (Certain problems of motion and stability of a heavy gyroscope). *Trudy Kazan. Aviats. Instituta* Vol. 38, 1958.

Translated by T.L.